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ON THE ACCRETION OF INHOMOGENEOUS VISCOELASTIC BODIES UNDER FINITE DEFORMATIONS*

V.V. METLOV

A formulation is given of the problem of the state of stress and strain of a continuously accreting body under finite deformations. The concept of a deformation gradient in an accreting body taking the deformation prehistory of the attached elements into account, is introduced. Boundary conditions are considered on the growth surface, which differ from the usual conditions in stresses or displacements. A solution is obtained for problems on the accretion of an inhomogeneous viscoelastic cylinder subjected to the preliminary deformation of the attached elements, to twisting, stretching, and imprinting loads, as well as on the simultaneous accretion and removal of a hollow sphere. Results of computations for the problem of winding a viscoelastic inhomogeneous cylinder with tension are presented.

1. Kinematics of an accreting body. Governing equation. By the accretion of a body we shall mean the continuous attachment of new elements to parts of its surface (the growth surface). The model of a continuously accreting body can describe the process of successive erection of structures, the gradual formation of a solid during a phase transition, the fabrication of bodies by means of winding, etc.

For fixed (non-accreting) bodies the reference description of the motion $\mathbf{x} = \boldsymbol{\chi}(t, \mathbf{X})$ is taken, where $\mathbf{x}$ is the radius-vector of a material point (particle) in an actual (running) configuration, and $\mathbf{X}$ is the radius-vector of this point in a fixed reference configuration. The governing equation is here written inthe fairly general form /1/

$$\mathbf{T}(t, \mathbf{X}) = \boldsymbol{\Phi}_{\varkappa}(t - t_0, \mathbf{F}^t(\mathbf{X}), \mathbf{X}), \ \mathbf{F}(\tau, \mathbf{X}) = \nabla_X \boldsymbol{\chi}(\tau, \mathbf{X}) \tag{1.1}$$

Here $\mathbf{T}$ is the Cauchy stress tensor, $\boldsymbol{\Phi}_{\varkappa}$ is a functional dependent on the chosen reference configuration $\varkappa$, and $\mathbf{F}^t(\mathbf{X}) = (\mathbf{F}^t(s, \mathbf{X}) = \mathbf{F}(t - s, \mathbf{X}), 0 \leqslant s < \infty)$ is the prehistory of the gradient $\mathbf{F}(\tau, \mathbf{X})$ of the deformation from the reference configuration $\varkappa$ into the actual configuration $\boldsymbol{\chi}(\tau)$ up to the running time t. The relationship (1.1) should satisfy the condition of invariance relative to the Galilean space-time transformation group /1/. Invariance relative to the shift of the time origin results in the fact that for a body with properties varying with time the functional $\boldsymbol{\Phi}_{\varkappa}$ can depend on the time t only by means of the difference $t - t_0$, where t_0 is a certain fixed time that can be taken as the time of body fabrication (generation). If different elements of an ageing body are fabricated at different times $t_0 = t_0(\mathbf{X})$, then it possesses an age inhomogeneity that was first investigated in /2/.

The feature of an accreting body is that it is impossible to fix any reference configuration of its particles since it is continuously supplemented by new particles. The purpose of this paper is to construct an analogue of the deformation gradient for an accreting body.

The triplet of numbers (τ^*, u_1, u_2) is the marker for particles being attached during accretion, where τ^* is the time of particle attachment to the body, and (u_1, u_2) are its curvilinear coordinates on the growth surface $S^*(\tau^*)$. In the general case, an arbitrary three-dimensional parameter ξ, mutually related one-to-one with the triplet (τ^*, u_1, u_2), can be taken as marker. Since different particles can be attached to the accreting body at different times at the very same point of space, or continuous attachement of particles in one site can occur (for a fixed growth surface), the particle radius-vector at the time of its attachment $\mathbf{X}(\xi)$ cannot generally be its marker.

Solving the equation $\mathbf{x} = \boldsymbol{\chi}(t, \xi)$ for ξ and substituting $\xi = \xi(t, \mathbf{x})$ into the particle velocity function $\boldsymbol{\chi}^{\cdot}(t, \xi)$, we obtain a spatial velocity field $\mathbf{v}(t, \mathbf{x})$ such that $\mathbf{v}(t, \boldsymbol{\chi}(t, \xi)) = \boldsymbol{\chi}^{\cdot}(t, \xi)$ (here and henceforth the dot denotes the partial derivative with respect to time). If the field $\mathbf{v}(t, \mathbf{x})$ is given, the motion $\boldsymbol{\chi}(t, \xi)$ is found by integrating the ordinary system $d\boldsymbol{\chi}/dt = \mathbf{v}(t, \boldsymbol{\chi}), \ \boldsymbol{\chi}(\tau^*) = \mathbf{X}(\xi)$.

The second-rank tensors are henceforth treated as linear amppings of a three-dimensional Euclidean vector space into itself /1/. The composition (matrix product) of the tensors $\mathbf{A}$

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and $\mathbf{B}$ is denoted by $\mathbf{AB}$, and the result of tensor $\mathbf{A}$ acting on the vector $\mathbf{a}$ by $\mathbf{Aa}$. The matrix of the components $A_k^{\ i}$ of the tensor $\mathbf{A}$ in some basis $(\mathbf{e}_k)_{k=1,2,3}$ is determined by the expansions $\mathbf{Ae}_k = A_k^{\ i}\mathbf{e}_i$ (summation is over repeated subscripts). The spatial argument $\mathbf{x}$ as well as ξ (respectively, X) is sometimes later omitted in the function. In equations containing functions of the variables $\mathbf{x}$ and ξ (respectively, $\mathbf{X}$) simultaneously, it is assumed that these variables are connected by the relationship $\mathbf{x} = \chi(t, \xi)$ (respectively, $\mathbf{x} = \chi(t, \mathbf{X})$).

The gradient $\mathbf{F}$ of a fixed body possesses the following properties.

1°. Let $\mathbf{Q}_\tau(t, \mathbf{X})$ be the deformation gradient when the body passes from the configuration $\chi(\tau)$ into the configuration $\chi(t)$. We denote the spatial velocity gradient by $G(t, \mathbf{X}) = \nabla_x \mathbf{v}(t, \mathbf{x})|_{\mathbf{x}=\chi(t,\mathbf{X})}$. The following relationships then hold ($\mathbf{I}$ is the unit tensor):

$$\mathbf{F}(t, \mathbf{X}) = \mathbf{Q}_\tau(t, \mathbf{X})\mathbf{F}(\tau, \mathbf{X}), \quad \mathbf{Q}_\tau(t, \mathbf{X}) = \int_\tau^t (\mathbf{I} + \mathbf{G}(\tau, \mathbf{X})\, d\tau) \tag{1.2}$$

The first formula is the rule of multiplication of gradients, and the second, which represents $Q_\tau(t)$ as a multiplicative integral /3/, is a consequence of the differential Eqs. /1/

$$\mathbf{F}^{\cdot}(t, \mathbf{X}) = \mathbf{G}(t, \mathbf{X})\mathbf{F}(t, \mathbf{X}), \quad \mathbf{Q}_\tau^{\cdot}(t, \mathbf{X}) = \mathbf{G}(t, \mathbf{X})\mathbf{Q}_\tau(t, \mathbf{X}) \tag{1.3}$$

2°. The density of the body mass in the actual configuration is $\rho(t, \mathbf{X}) = \rho_\times(\mathbf{X})/\det \mathbf{F}(t, \mathbf{X})$, where $\rho_\times$ is the density of the mass in the reference configuration; in particular $\rho(t, \mathbf{X}) = \rho(\tau, \mathbf{X})/\det \mathbf{Q}_\tau(t, \mathbf{X})$. Hence, and from (1.3) we obtain the equation of continuity /1/: $\rho^{\cdot} + \rho \operatorname{div}_x \mathbf{v} = 0$.

3°. On substituting a reference system generating an orthogonal transformation of space with the tensor $\mathbf{O}(t)$, the deformation gradient is transformed by means of the formula /1/ $\mathbf{F}^*(t) = \mathbf{O}(t)\mathbf{F}(t)$.

At any time τ satisfying the condition $\tau > \tau^*(\xi)$, the accreting body contains the point ξ together with its neighbourhood. Therefore, the deformation gradient $\mathbf{Q}_\tau(t, \xi)$ can be determined when the neighbourhood of the point ξ passes from the configuration $\chi(\tau)$ into $\chi(t)$ for any $t \geqslant \tau > \tau^*(\xi)$. We will here have

$$\mathbf{Q}_\tau^{\cdot}(t, \xi) = \mathbf{G}(t, \xi)\mathbf{Q}_\tau(t, \xi), \quad \mathbf{Q}_\tau(t, \xi) = \int_\tau^t (\mathbf{I} + \mathbf{G}(\tau, \xi)\, d\tau), \quad \mathbf{G}(\tau, \xi) = \nabla_x \mathbf{v}(\tau, \mathbf{x}) \tag{1.4}$$

Extending the function $\mathbf{Q}_\tau(t, \xi)$ in continuity to the point $\tau = \tau^*(\xi)$, we obtain $\mathbf{Q}(t, \xi) = \lim \mathbf{Q}_\tau(t, \xi)$ (as $\tau \to \tau^*(\xi) + 0$), the gradient of the deformation when a body element passes from the initial state into the actual state at the time of attachment. At the time of attachment $\tau^*(\xi)$, a body element in the neighbourhood of the point ξ can have a deformation gradient $\mathbf{F}_*(\xi)$ different from $\mathbf{I}$ with respect to a certain reference state because of the prehistory of its deformation. Using the gradient multiplication rule, we obtain for the deformation gradient during element passage from the reference to the actual state

$$\mathbf{F}(t, \xi) = \mathbf{Q}(t, \xi)\mathbf{F}_*(\xi), \quad \mathbf{Q}(t, \xi) = \int_{\tau^*(\xi)}^t (\mathbf{I} - \mathbf{G}(\tau, \xi)\, d\tau) \tag{1.5}$$

The gradient $\mathbf{F}(t, \xi)$, defined by (1.5), possesses the properties 1° – 3° on replacing $\mathbf{X}$ by ξ and $\rho_\times(\mathbf{X})$ by $\rho_0(\xi)$, the mass density of body elements inthe reference state. Using the functional Φ that satisfies the condition of independence from the reference system /1/, we can write the governing equation for the accreting system in the form

$$\mathbf{T}(t, \xi) = \Phi(t - t_0(\xi), \mathbf{F}^t(\xi), \xi), \quad t \geqslant t_0(\xi), \quad t_0(\xi) \leqslant \tau^*(\xi) \tag{1.6}$$

The initial state of stress $\mathbf{T}^*(\xi) = \Phi(\tau^* - t_0, \mathbf{F}^{\tau^*}, \xi)$ satisfies the equilibrium condition with external force action on the growth surface $S^*(\tau^*)$ with normal $\mathbf{n}$

$$\mathbf{T}^*(\xi)\mathbf{n} = \mathbf{P}(\tau^*(\xi), \mathbf{x})|_{\mathbf{x} \in S^*(\tau^*)} \tag{1.7}$$

where $\mathbf{P}(t, \mathbf{x})$ is the stress vector acting on $S^*(t)$. The remaining three components of the initial stress tensor $\mathbf{T}^*(\xi)$ as well as the tensor $\mathbf{F}_*(\xi) = \mathbf{F}(\tau^*(\xi)\, \xi)$ are determined by the prehistory $\mathbf{F}^{\tau^*}$ of the element deformation in the neighbourhood of the particle ξ /4/.

Let us present the formula determining $\mathbf{Q}(t, \xi)$ for two particular kinds of deformation. Suppose the tensors $\mathbf{G}(\tau, \xi)$ commute for all $\tau \geqslant \tau^*(\xi)$. Then

$$\mathbf{Q}(t, \xi) = \exp\left\{\int_{\tau^*(\xi)}^t \mathbf{G}(\tau, \xi)\, d\tau\right\} \tag{1.8}$$

Let the initial radius-vector $\mathbf{X}(\xi)$ be one-to-one functions of ξ so that the motion can be represented as a function of $\mathbf{x} = \chi(t, \mathbf{X})$. Then

$$\mathbf{Q}(t,\mathbf{X}) = \mathbf{F}_1(t,\mathbf{X})\mathbf{F}_1^{-1}(\tau^*(\mathbf{X}),\mathbf{X}),\quad \mathbf{F}_1(t,\mathbf{X}) = \nabla_X \chi(t,\mathbf{X}) \tag{1.9}$$

2. Kinematics formulas in an accompanying basis. Let $(\xi^i)_{i=1,2,3}$ be a set of marker coordinates $\xi = \xi(\tau^*, u_1, u_2)$. As in /5, 6/, we introduce the basis vectors $\mathbf{e}_i(t,\xi) = \partial\chi(t,\xi)/\partial\xi^i$, $t > \tau^*(\xi)$. We set

$$\mathbf{e}_i^*(\xi) = \lim_{\tau\to\tau^*(\xi)+0} \mathbf{e}_i(\tau,\xi),\quad \mathbf{e}_i^\circ(\xi) = \mathbf{F}_*^{-1}(\xi)\mathbf{e}_i^*(\xi),\quad g_{ij} = (\mathbf{e}_i\cdot\mathbf{e}_j)$$

$$g_{ij}^* = (\mathbf{e}_i^*\cdot\mathbf{e}_j^*),\quad g_{ij}^\circ = (\mathbf{e}_i^\circ\cdot\mathbf{e}_j^\circ),\quad \varepsilon_{ij}(t,\xi) = (g_{ij} - g_{ij}^\circ)/2$$

where $(\mathbf{a}\cdot\mathbf{b})$ is the scalar product. We then have

$$\mathbf{e}_i(t,\xi) = \mathbf{Q}(t,\xi)\mathbf{e}_i^*(\xi) = \mathbf{F}(t,\xi)\mathbf{e}_i^\circ(\xi),\quad \mathbf{Q}(t,\xi) = \mathbf{e}_i(t,\xi)\otimes\mathbf{e}^{*i}(\xi) \tag{2.1}$$

$$\mathbf{E}_1 = \varepsilon_{ij}\mathbf{e}^{\circ i}\otimes\mathbf{e}^{\circ j} = {}^1/_2(\mathbf{C}-\mathbf{I}),\quad \mathbf{C} = \mathbf{F}^T\mathbf{F},\quad \mathbf{F} = \mathbf{e}_i\otimes\mathbf{e}^{\circ i}$$

$$\mathbf{E}_2 = \varepsilon_{ij}\mathbf{e}^i\otimes\mathbf{e}^j = {}^1/_2(\mathbf{I}-\mathbf{B}^{-1}),\quad \mathbf{B} = \mathbf{F}\mathbf{F}^T$$

$$\varepsilon_{ij}(t,\xi) = \varepsilon_{ij}^*(\xi) + \int_{\tau^*(\xi)}^{t} e_{ij}(\tau,\xi)\,d\tau,\quad \varepsilon_{ij}^* = \frac{1}{2}(g_{ij}^* - g_{ij}^\circ)$$

$$e_{ij}(t,\xi) = {}^1/_2(v_{i,j} + v_{j,i}),\quad v_{i,j} = (\mathbf{e}_i\cdot\mathbf{G}\mathbf{e}_j)$$

Here $\mathbf{a}\otimes\mathbf{b}$ is the tensor (dyadic) product of the vectors $\mathbf{a}$ and $\mathbf{b}$ /1, 5, 6/, $\mathbf{E}_1$ and $\mathbf{E}_2$ are the first and second finite deformation tensors /6/, $\mathbf{F}^T$ is the transpose of the tensor $\mathbf{F}$. e_{ij} and $v_{i,j}$ are the strain rate tensor components /5/, $\mathbf{D} = (\mathbf{G}+\mathbf{G}^T)/2$, and the spatial velocity gradient in the basis $\mathbf{e}^i\otimes\mathbf{e}^j(t,\xi)$. $(\mathbf{e}^{\circ i})$. $(\mathbf{e}^{*i})$ and $(\mathbf{e}^i)$ are bases reciprocal to $(\mathbf{e}_i^\circ)$, $(\mathbf{e}_i^*)$ and $(\mathbf{e}_i)$. and $\mathbf{A}^{-1}$ is the inverse tensor to the tensor $\mathbf{A}$.

3. Accretion law. Formulation of the boundary value problem. The process of new element attachment to the accreting body is characterized by the flux density vector of the substance being attached $\mathbf{J}(t,\mathbf{x})$ on the growth surface $S^*(t)$. Considering the mass balance on the growth surface, we obtain the condition

$$J_n + \rho^*(s_n - v_n)|_{S^*(t)} = 0,\quad \rho^*(\xi) = \frac{\rho_0(\xi)}{\det\mathbf{F}_*(\xi)},\quad s_n = \left(\frac{\partial\mathbf{X}(\xi)}{\partial\tau^*}\cdot\mathbf{n}\right) \tag{3.1}$$

Here ρ^* is the volume mass density of the elements being attached $J_n < 0$ and v_n are the projections of the vector $\mathbf{J}$ and the material velocity $\mathbf{v}$ on the external normal $\mathbf{n}$ to S^*. and s_n is the velocity of the motion of the surface S^* along the normal $\mathbf{n}$.

At the time $t = 0$ let accretion of the initial body occupying the domain Ω_0 start on part of its surface $S^*(0)$. Stress and displacement vectors, respectively, are given on other parts of its surface $S_\sigma(t)$ and $S_\chi(t)$. The flux vector $\mathbf{J}(t,\mathbf{x})$ or the flux component $J_n < 0$ along the normal to $S^*(t)$ is given in formulating the boundary value problem on $S^*(t)$; furthermore, the value $\rho_0(\xi)$ of the density, the magnitude $\mathbf{P}(t,\mathbf{x})$ and the prehistory of the gradient $\mathbf{F}^{\tau^*} = (\mathbf{F}(\tau,\xi).\ \tau\in[t_0(\xi).\ \tau^*(\xi)]$. $\mathbf{F}(\tau^*(\xi).\ \xi) = \mathbf{F}_*(\xi)$. that satisfies condition (1.7) as well as the vector $\mathbf{b}(t,\mathbf{x})$ of the mass force density are given. It is required to determine the law of variation of the domain $\Omega(t)$ occupied by the body, and the deformation $\chi(t,\xi)$ satisfying the boundary condition (3.1) and the quasistatic equilibrium equation $\operatorname{div}_x\mathbf{T} + \rho\mathbf{b} = 0$, where $\mathbf{T}$ is defined by the relationship (1.6) and the density is $\rho(t,\xi) = \rho_0(\xi)\det\mathbf{F}(t,\xi)$. It is simultaneously necessary to find the deformation $\chi_0(t,\mathbf{X}_0)$ of the points $\mathbf{X}_0\in\Omega_0$ of the initial body that satisfies the equilibrium equation, the boundary conditions on $S_\sigma(t)$ and $S_\chi(t)$, and the conditions of connection with the grown part of the body.

4. On accretion of a viscoelastic hollow incompressible cylinder. Let (R_0,Θ_0,Z_0) be cylindrical coordinates in the initial configuration of the original hollow cylinder $\Omega_0 = \{(R_0,\Theta_0,Z_0): R_1 \leqslant R_0 \leqslant R_2\}$. Starting with the time $t = 0$ its outer accretion occurs with a radial flux vector of magnitude $\rho J_0(t)$, where ρ is a constant density. The internal pressure $p_i(t)$ and external pressure $p_e(t)$, the axial force $P_z(t)$ and the torque $M_z(t)$, that change continuously for zero initial values, start to act simultaneously. Let (r,φ,z) be cylindrical coordinates in space, and $c(t)$ the external radius of the cylinder. We seek the spatial velocity field in the form

$$v_r = f(t,r),\ v_\varphi = r\psi(t)z,\ v_z = \alpha(t)z \tag{4.1}$$

The incompressibility condition $\operatorname{div}_x\mathbf{v}(t,\mathbf{x})$ yields $\partial f/\partial r + f/r + \alpha = 0$, from which we find ($\beta(t)$ is an undetermined function)

$$f(t,r) = -\alpha(t)r/2 + \beta(t)/r \tag{4.2}$$

Let $r_0(t,R_0)$, $\varphi_0(t,\Theta_0,Z_0)$, $z_0(t,Z_0)$ be cylindrical coordinates of the running radius vector $\mathbf{x} = \chi_0(t,\mathbf{X}_0)$ of the point $\mathbf{X}_0(R_0,\Theta_0,Z_0)$ of the original cylinder $r(t,\tau^*)$, $\varphi(t,\tau^*,\Theta,Z)$. $z(t,\tau^*,Z)$

are cylindrical coordinates of the running radius-vector $\mathbf{x} = \boldsymbol{\chi}(t, \xi)$ of the point $\xi = (\tau^*, \Theta, Z)$ of the grown part of the cylinder, where (Θ, Z) are particle coordinates at the time of its attachment τ^* to the growth surface $S^*(\tau^*) = \{(r, \varphi, z) : r = c(\tau^*)\}$. Integrating the ordinary system corresponding to (4.1), and taking account of (4.2) we obtain

$$r_0^2(t, R_0) = R_0^2 a(t, 0) + \int_0^t a(t, \tau)\, 2\beta(\tau)\, d\tau \tag{4.3}$$

$$a(t, \tau) = \exp\left\{-\int_\tau^t \alpha(s)\, ds\right\}$$

$$r^2(t, \tau^*) = c^2(\tau^*) a(t, \tau^*) + \int_{\tau^*}^t a(t, \tau)\, 2\beta(\tau)\, d\tau$$

$$\varphi_0(t, \Theta_0, Z_0) = \Theta_0 + \int_0^t z_0(\tau, Z_0)\psi(\tau)\, d\tau, \quad z_0(t, Z_0) = \frac{Z_0}{a(t, 0)}$$

$$\varphi(t, \tau^*, \Theta, Z) = \Theta + \int_{\tau^*}^t \psi(\tau) z(\tau, \tau^*, Z)\, d\tau, \quad z(t, \tau^*, Z) = \frac{Z}{a(t, \tau^*)}$$

We use the relationships /1/

$$\nabla_X \mathbf{x}(t, \mathbf{X}) = \frac{\partial x^k}{\partial X^l} \mathbf{e}_k \otimes \mathbf{h}^l, \quad \mathbf{e}_k = \frac{\partial \mathbf{x}}{\partial x^k} \tag{4.4}$$

$$\mathbf{h}_i = \frac{\partial \mathbf{X}}{\partial X^i}, \quad A_{kl} = A_l^{\,k} |\mathbf{e}_k| \cdot |\mathbf{h}^l|$$

in calculating the deformation gradient, where $x^k(\mathbf{x})$ and $X^l(\mathbf{X})$ $(k, l = 1, 2, 3)$ are arbitrary coordinate systems in the actual and reference configurations, $(\mathbf{h}_i \cdot \mathbf{h}^l) = \delta_i^{\,l}$, and A_{kl} are physical components of the tensor $\mathbf{A} = A^k{}_l \mathbf{e}_k \otimes \mathbf{h}^l$ in the orthogonal curvilinear coordinate systems $(x^k(\mathbf{x}))$ and $(X^l(\mathbf{X}))$.

For points of the grown domain it is first necessary to express the coordinates at the time t in terms of coordinates at the time $\tau > \tau^*$. The corresponding formulas have the form

$$r^2(t) = r^2(\tau) a(t, \tau) + \int_\tau^t a(t, s)\, 2\beta(s)\, ds \tag{4.5}$$

$$\varphi(t) = \varphi(\tau) + \int_\tau^t \psi(s) z(s)\, ds, \quad z(t) = z(\tau)/a(t, \tau)$$

Let $\mathbf{e}_i(\mathbf{x})$ and $\mathbf{e}_i(\mathbf{X}_0)$ denote unit vectors of the basis in the cylindrical coordinate system in the actual and initial configurations $\mathbf{e}_i^*(\xi) = \mathbf{e}_i(\mathbf{X}(\xi))$, $i = 1, 2, 3$, $\mathbf{X}(\xi) = \boldsymbol{\chi}(\tau^*(\xi), \xi)$. Evaluating the tensors $\mathbf{F}_0(t, \mathbf{X}_0) = \nabla_X \boldsymbol{\chi}_0(t, \mathbf{X}_0)$ and $\mathbf{Q}_\tau(t, \xi)$ by means of (4.3)-(4.5) and letting $\tau \to \tau^*$, we have for the non-zero physical components of the tensors $\mathbf{F}_0(t, \mathbf{X}_0) = F_{ij}^{\circ}(t, R_0)\mathbf{e}_i(\mathbf{x}) \otimes \mathbf{e}_j(\mathbf{X}_0)$ and $\mathbf{Q}(t, \xi) = Q_{ij}(t, \tau^*)\mathbf{e}_i(\mathbf{x}) \otimes \mathbf{e}_j^*(\xi)$ the following relationships:

$$F_{11}^{\circ} = (A_0 B_0)^{-1}, \; Q_{11} = (A_1 B_1)^{-1}, \; F_{22}^{\circ} = B_0, \; Q_{22} = B_1 \tag{4.6}$$

$$F_{23}^{\circ} = C_0, \; Q_{23} = C_1, \; F_{33}^{\circ} = A_0, \; Q_{33} = A_1$$

$$A_0 = a(t, 0)^{-1}, \; A_1 = a(t, \tau^*)^{-1}$$

$$B_0 = r_0(t, R_0)/R_0, \; B_1 = r(t, \tau^*)/c(\tau^*)$$

$$C_0 = r_0(t, R_0)\int_0^t a(\tau, 0)^{-1}\psi(\tau)\, d\tau, \quad C_1 = r(t, \tau^*)\int_{\tau^*}^t a(\tau, \tau^*)^{-1}\psi(\tau)\, d\tau$$

Let the gradient prehistories $(\mathbf{F}(\tau, \xi), t_0(\xi) \leqslant \tau \leqslant \tau^*)$ satisfy the relations $t_0 = t_0(\tau^*)$, $\mathbf{F} = F_{ij}(\tau, \tau^*)\mathbf{g}_i(\tau, \xi) \otimes \mathbf{g}_j^{\circ}(\xi)$, where $(\mathbf{g}_i(\tau, \xi))$ is an orthonormalized basis such that $\mathbf{g}_i(\tau^*, \xi) = \mathbf{e}_i^*(\xi)$, $\mathbf{g}_j^{\circ}(\xi) = \mathbf{g}_i(t_0, \xi)$. The matrix of the components $F_{ij}(\tau, \tau^*)$ here has, for $\tau \in [t_0, \tau^*]$, the structure (4.6) with parameters A_2, B_2, C_2 dependent on τ and τ^*. A continuous model of the accretion of thin-walled tubes experiencing torsion by preliminary deformation, axial tension, and impression /1/ can be an illustration of the deformation prehistory considered (compare with (4.3) and (4.6))

$$r = \sqrt{R^2/e(t) + b(t)}, \quad \varphi = \Theta + \Psi(t) Z, \quad z = e(t) Z \tag{4.7}$$

The total deformation gradient for $t \geqslant \tau^*(\xi)$ will be $(\mathbf{F}_*(\xi) = F_{kl}^*(\tau^*)\mathbf{e}_k^*(\xi) \otimes \mathbf{g}_l^{\circ}(\xi))$

$$\mathbf{F}(t, \xi) = \mathbf{Q}(t, \xi)\mathbf{F}_*(\xi) = Q_{ij}(t, \tau^*) F_{jl}^*(\tau^*)\mathbf{e}_i(\mathbf{x}) \otimes \mathbf{g}_l^{\circ}(\xi) = F_{il}(t, \tau^*)\mathbf{e}_i(\mathbf{x}) \otimes \mathbf{g}_l^{\circ}(\xi) \tag{4.8}$$

Multiplying the matrices (4.6) according to (4.8) we obtain the following expressions for the non-zero components $F_{ij}(t, \tau^*)$ $(t \geqslant \tau^*)$

$$F_{11} = (AB)^{-1}, \; F_{22} = B, \; F_{23} = C, \; F_{33} = A \tag{4.9}$$

$$A = A^* A_1, \; B = B^* B_1, \; C = C^* B_1 + A^* C_1$$

$$A^* = A_2(\tau^*, \tau^*),\ B^* = B_2(\tau^*, \tau^*),\ C^* = C_2(\tau^*, \tau^*)$$

We take the viscoelastic analogue of the equation for a neo-Hookean body proposed for rubberlike filled polymers* (*Adamov, A.A. Construction of the equation of state of a viscoelastic weakly-compressible material under finite deformations. Candidate dissertation. Perm, Ural Science Centre, USSR Academy of Sciences, Institute of Mechanics of Continuous Media, 1979.)

$$\mathbf{T} = -p\mathbf{I} + \mathbf{F}(t) L(t - t_0, [\mathbf{I} - \mathbf{C}^{-1} \operatorname{tr} \mathbf{C}/3]^t) \mathbf{F}^T(t), \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} \tag{4.10}$$

as the governing equation.

Here p is the hydrostatic pressure, L is a linear functional acting according to the rule $(\varphi^t(s) = \varphi(t - s))$:

$$L(t - t_0, \varphi^t) = \int_0^{t-t_0} \mu(t - t_0, t - t_0 - s)\, d_s\varphi^t(s) = \int_{t_0}^{t} \mu(t - t_0, \tau - t_0)\, d_\tau\varphi(\tau) \tag{4.11}$$

For small deformations (4.10) becomes the governing equation of linear viscoelasticity $\mathbf{T} = -p\mathbf{I} + 2L(t - t_0, [\mathbf{E} - \mathbf{I} \operatorname{tr} \mathbf{E}/3]^t)$, where $\mathbf{E}$ is a tensor of infinitesimal deformations. Therefore, if the body is subjected to the governing Eq. (4.10), then $\mu(t - t_0, \tau - t_0)$ is the relaxation function for small shear deformations.

The tensor $\mathbf{C}(\tau, \xi)$ for $t_0 \leqslant \tau \leqslant \tau^*$ equals

$$\begin{aligned} \mathbf{C}(\tau, \xi) &= (F_{ij}(\tau, \tau^*) \mathbf{g}_j^\circ(\xi) \otimes \mathbf{g}_i(\tau, \xi))(F_{kl}(\tau, \tau^*) \mathbf{g}_k(\tau, \xi) \otimes \mathbf{g}_l^\circ(\xi)) = \\ &F_{ij}(\tau, \tau^*) F_{il}(\tau, \tau^*) \mathbf{g}_j^\circ(\xi) \otimes \mathbf{g}_l^\circ(\xi) \end{aligned} \tag{4.12}$$

We similarly obtain

$$\mathbf{C}(t, \xi) = \mathbf{g}_i^\circ(\xi) \otimes \mathbf{g}_j^\circ(\xi) F_{ki}(t, \tau^*) F_{kj}(t, \tau^*), \quad t \geqslant \tau^* \tag{4.13}$$

According to (4.12) and (4.13), the non-zero components of the tensor $\mathbf{C}(t, \xi)$ are the following for all $t \geqslant t_0(\tau^*)$ in the basis $\mathbf{g}_i^\circ(\xi) \otimes \mathbf{g}_j^\circ(\xi)$:

$$C_{11} = (AB)^{-2},\ C_{22} = B^2,\ C_{23} = C_{32} = BC,\ C_{33} = C^2 + A^2 \tag{4.14}$$

Here all the components are functions of t and τ^* and for $t \leqslant \tau^*$ we have $A = A_2$, $B = B_2$, $C = C_2$. According to (4.10) and (4.14), the non-zero components of the tensor $\mathbf{H} = \mathbf{T} + p\mathbf{I}$ are equal in the basis $\mathbf{e}_i(\mathbf{x}) \otimes \mathbf{e}_j(\mathbf{x})$, to

$$\begin{aligned} H_{11} &= (AB)^{-2} L(t - t_0, [1 - (AB)^2 I_1]^t) \\ H_{22} &= B^2 L(t - t_0, [1 - ((C/AB)^2 + B^{-2}) I_1]^t) - \\ &\quad (C^2 - 2BC) L(t - t_0, Q^t),\ Q = A^{-2}B^{-1}CI_1 \\ H_{23} &= H_{32} = -ABL(t - t_0, Q^t) + ACL(t - t_0, [1 - A^{-2}I_1]^t) \\ H_{33} &= A^2 L(t - t_0, [1 - A^{-2}I_1]^t),\ I_1 = [(AB)^{-2} + B^2 + C^2 + A^2]/3 \end{aligned} \tag{4.15}$$

The components of the tensor $\mathbf{H}$ in the basis $\mathbf{e}_i(\mathbf{x}) \otimes \mathbf{e}_j(\mathbf{x})$ have the form (4.15) also for points of the original cylinder if A, B, C are replaced by A_0, B_0, C_0. The time t_0 can depend on R_0 for points of the original cylinder (age inhomogeneity); the kernel μ can also contain an explicit dependence on R_0 (respectively, on τ^* for points of the grown part), i.e., $\mu = \mu(t, \tau, R_0)$ (respectively, $\mu = \mu(t, \tau, \tau^*)$). Under these conditions the components of the tensor $\mathbf{H}$ depend only on t and R_0 (t and τ^*, respectively). From the relationships (4.3) we find the inverse functions $R_0 = R_0(t, r)$ and $\tau^* = \tau^*(t, r)$; by substituting them into the functions $H_{ij}(t, R_0)$ and $H_{ij}(t, \tau^*)$ we obtain that $H_{ij} = H_{ij}(t, r)$. Hence, and from the equilibrium equation $\operatorname{div}_\mathbf{x}(\mathbf{H} - p\mathbf{I}) = 0$, we will have

$$\begin{aligned} &\partial_r\sigma_r - \sigma/r = 0,\ \partial_\varphi p = 0,\ \partial_z p = 0 \\ &\sigma_r = T_{11},\ \sigma_\varphi = T_{22},\ \sigma_z = T_{33} \\ &\sigma = \sigma_\varphi - \sigma_r,\quad \mathbf{T} = T_{ij}\mathbf{e}_i(\mathbf{x}) \otimes \mathbf{e}_j(\mathbf{x}) \end{aligned} \tag{4.16}$$

and analogous equalities for points of the original cylinder. Hence

$$\begin{aligned} &p = p(t, r),\ T_{ij} = T_{ij}(t, r) \\ &\sigma_r^\circ(t, r) = -p_i(t) + V^\circ(t, r) - V^\circ(t, r_0(t, R_1)) \\ &r_0(t, R_1) \leqslant r \leqslant r_0(t, R_2) \\ &\sigma_r(t, r) = -p_e(t) + V(t, r) - V(t, c(t)), \\ &r(t, 0) \leqslant r \leqslant c(t) \\ &V^\circ(t, r) = \int \frac{\sigma^\circ(t, r)\, dr}{r},\quad V(t, r) = \int \frac{\sigma(t, r)\, dr}{r} \end{aligned} \tag{4.17}$$

where σ_r°, σ° refer to points of the original cylinder.

From the governing Eq. (4.10) we have

$$\sigma = H_{22} - H_{11}, \ \sigma_z - \sigma_r = H_{33} - H_{11} \qquad (4.18)$$
$$T_{23} = T_{32} = H_{23}, \ T_{12} = T_{13} = 0$$

and analogous equalities for points of the original cylinder. Therefore, the deformations and stresses are expressed according to (4.3), (4.6), (4.9), (4.15)-(4.18) in terms of the undetermined functions of the time $\alpha(t), \beta(t), \psi(t)$ and $c(t)$. The system of equations to determine them consists of the continuity condition for the radial stress on the boundary of the initial and grown domains (see (4.17))

$$V^\circ(t, r_0(t, R_2)) - V^\circ(t, r_0(t, R_1)) + V(t, c(t)) - V(t, r(t, 0)) = (p_i - p_e)(t) \qquad (4.19)$$

the equilibrium equations of the cylinder endfaces

$$2\pi \int_{r_0(t, R_1)}^{c(t)} \sigma_z(t, r) r\, dr = P_z(t) \qquad (4.20)$$
$$2\pi \int_{r_0(t, R_1)}^{c(t)} T_{23}(t, r) r^2\, dr = M_z(t)$$

(where the zero superscript has been omitted, for brevity, on the functions of points of the original domain), and the accretion condition (3.1)

$$dc/dt = J_0(t) - \alpha(t) c(t)/2 + \beta(t)/c(t), \ J_0(t) > 0 \qquad (4.21)$$

For a numerical example, we set $M_z \equiv 0$ and $\alpha(t) \equiv 0$ (plane deformation). Let $p_i(t) \equiv 0$ and $p_e(t) \equiv 0$. Therefore, there are no external loads, except P_z. If there was no preliminary deformation here, i.e., $\mathbf{F}(\tau, \xi) \equiv \mathbf{I}$ for $t_0 \leqslant \tau \leqslant \tau^*$, then there would be no stress (and $P_z \equiv 0$). Let $t_0(\tau^*) = \tau^* - 0$, and the preliminary deformation will be the elastic instantaneous deformation at time τ^* of the form

$$\mathbf{F}_*(\xi) = F_*^{-1} \mathbf{e}_1^* \otimes \mathbf{e}_1^* + F_* \mathbf{e}_2^* \otimes \mathbf{e}_2^* + \mathbf{e}_3^* \otimes \mathbf{e}_3^*, \quad F_* = F_*(\tau^*)$$

The formulation of the problem under consideration can be the model of a cylinder accretion process by means of winding with tension. The magnitude of the force σ_φ^* during winding is determined from (4.10): $\sigma_\varphi^* = G(0)(F_*^2 - F_*^{-2})$, where $G(t) = \mu(t, t)$ is the shear modulus.

The problem reduces to determining the functions $\beta(t)$ and $c(t)$ from the system (4.19), (4.21). We make the change of variable $r = r_0(t, R_0)$ in the integral over the original cylinder in (4.19), and $r = r(t, s)$ in the integral over the grown part, where according to (4.3)

$$r_0^2(t, R_0) = R_0^2 + 2b(t), \quad r^2(t, s) = c^2(s) + 2(b(t) - b(s)), \quad b(t) = \int \beta(t)\, dt, \quad b(0) = 0.$$

After changing the order of integration, Eq. (4.19) in the presence of just the age inhomogeneity in the grown part will take the form $(t_0(R_0) \equiv 0)$

$$\int_0^t K_1(t, \tau, b^\tau, c^\tau, b(t)) \beta(\tau)\, d\tau + \int_0^t K_2(t, \tau, c(\tau), b(t))\, d\tau = 0 \qquad (4.22)$$

$$K_1 = \sum_{i=1}^{3} K_{1i}, \quad K_{11} = \frac{1}{3} \mu(t, \tau) \varphi(y) \Big|_{R_2^2}^{R_1^2}$$

$$\varphi(y) = \frac{[3 - 2b(\tau)/b(t)}{y + 2b(t)} + \frac{b(\tau) \ln[1 + 2b(t)/y]}{b^2(t)} + \frac{3y + 4b(\tau)}{(y + 2b(\tau))^2}$$

$$K_{12} = \frac{2}{3} \int_0^{\tau \wedge T} \frac{\mu_1(t, \tau, s)(1 + 2w^{-1}(\tau, s))}{r^4(t, s)}\, ds$$

$$K_{13} = \frac{2}{3} \int_0^{\tau \wedge T} \frac{\mu_1(t, \tau, s)(1 + 2w(\tau, s))}{r^4(t, s)}\, ds$$

$$\mu_1(t, \tau, s) = \mu(t - s, \tau - s) J_0(s) c(s), \quad w(\tau, s) = r^2(\tau, s) F_*^2(s) / c^2(s),$$
$$\tau \wedge T = \min(\tau, T)$$
$$K_2 = {}^1/_3\, \mu_1(t, \tau, \tau)[Q^2(2F_*^2 - F_*^{-2} - 1) - Q^{-2}(2F_*^{-2} - F_*^2 - 1)] / r^2(t, \tau)$$
$$Q = r(t, \tau) / c(\tau), \quad F_* = F_*(\tau)$$

where T is the time of growth termination.

After discretization of the time, system (4.21) and (4.22) was solved by a step-by-step method. Eq. (4.21) was integrated by Euler's method; the integrals in (4.22) were evaluated by the rectangle formula. The non-linear equation in the quantity $\beta_j = \beta(t_j)$, $t_j = j\Delta t$, where Δt is the integration spacing, was solved at the j-th step by successive approximations, where we have for the $(n+1)$-th approximation $b^{(n+1)}(t_j) = b(t_{j-1}) + \beta_j^{(n)} \Delta t$, $\beta_j^{(0)} = 0$. The correctness and accuracy of the calculations were checked by verifying the equilibrium Eq. (4.17). The modulus G was assumed to be constant in the computations, and the kernel μ is given by the formula /7/ (time is in conditional units)

$$\mu(t,\tau)G^{-1} = 1 - \varphi(\tau)\{1 - \exp[-\gamma(t-\tau)]\}$$
$$\varphi(\tau) = C + A\exp(-\beta_1\tau)$$
$$C = 0.05,\quad A = 0.75,\quad \beta_1 = 0.01,\quad \gamma = 0.1$$

The ratio of the characteristic ageing time β^{-1} and the relaxation time γ^{-1} is in the tens, which corresponds to data on polymer ageing /8/. The stresses are measured in units of G and the distance in units of the inner radius R_1. Let us give the following parameters: $R_2 = 1.2$; $F_* = 1.2$; $J_0T = 0.8$. We have thereby fixed the sequence of raising and lowering /9/. We show in Fig. 1 the time dependences of the functions $c(t)$ and $b(t)$ for $T=100$ (solid line) and $T = 20$ (dashes); the dash-dot line is the dependence of the outer radius $c(t)$ on time in the absence of tension $(F_* = 1)$. Diagrams of the stress $\sigma = |\sigma_\varphi| - |\sigma_r|$, respectively, at the time of growth termination $t = T$ and at the time $t = T + 50$ (the steady-state value of the residual stress), for $T = 100$ (solid line), and for $T = 20$ (dashes) are shown in Figs. 2 and 3.

Growth of the compressive stress in the original cylinder occurs during accretion because of the pressure of the pre-stretched elements; the initial stress $\sigma(\tau^*) = 0.746$ in the attached elements relaxes simultaneously. For $T = 100$ the relaxation of the tensile stress in the attached elements is developed to a greater degree than for $T = 20$ (Fig. 2), which results in a lower pressure on the original cylinder and less intensive creep (Fig. 1). After termination of the accretion, the creep rate changes sign and relaxation of the residual stresses occurs at all points of the cylinder. For $T = 100$ the cylinder is stiffer (because of ageing) and more inhomogeneous than for $T = 20$. Consequently, the process of residual stress relaxation for $T = 20$ is very much more intensive than for $T = 100$. Hence, although the compressive stresses at the end of the accretion are twice as great for $T = 20$ than for $T = 100$ (Fig. 2), the residual stresses being relaxed at $T = 20$ are less than at $T = 100$ (Fig. 3).

5. Accretion and reduction of a sphere.

We shall understand reduction to be the opposite of accretion, i.e., a continuous decrease in the mass of a body because of removal of elements from parts of its surface (reduction surface). The reduction process can be given by condition (3.1) on the reduction surface where $J_n > 0$, $\mathbf{J}$ is the flux density of the substance being removed from the surface $\partial\mathbf{X}/\partial\tau^*$ is replaced by $\partial\mathbf{x}^\circ/\partial t$, $\mathbf{x}^c(t, u_1, u_2)$ is the radius-vector of points of space that belong to the reduction surface $S^\circ(t)$, (u_1, u_2) are curvilinear coordinates on this surface, and $\rho^*(\xi)$ is replaced by $\rho(t,\mathbf{X}) = \rho_\varkappa(\mathbf{X})/\det\mathbf{F}(t,\mathbf{X})$. The kinematics in a decreasing body and the boundary conditions for the stresses are the same as for a fixed body.

Let (R, Θ, Λ) be spherical coordinates in the initial configuration of an original hollow incompressible sphere $\Omega_0 = \{(R,\Theta,\Lambda): R_1 \leqslant R \leqslant R_2\}$. At the time $t = 0$ simultaneous reduction of the sphere from within and accretion from outside starts with radial vectors of the flux of the magnitude $\rho J_i(t)$ and $\rho J_e(t)$, respectively (ρ is a constant density). Internal and

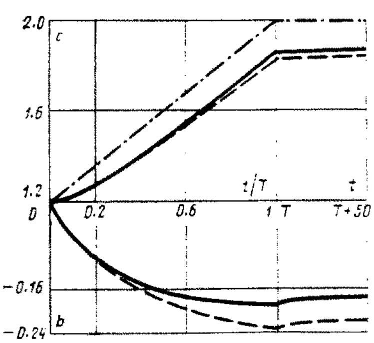

Fig.1

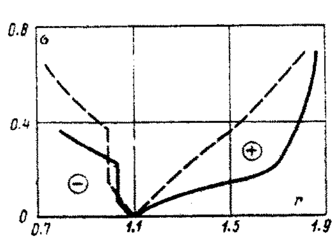

Fig. 2

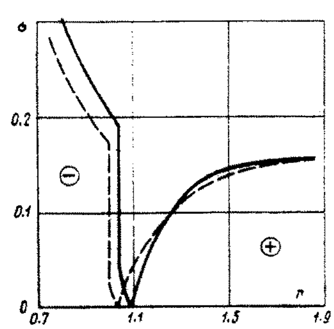

Fig. 3

external pressures $p_i(t)$ and $p_e(t)$ that change continuously under zero initial conditions, act here. The spatial velocity field in the spherical coordinates (r, φ, λ) will be

$$v_r = f(t, r),\ v_\varphi = 0,\ v_\lambda = 0 \tag{5.1}$$

The incompressibility condition $\operatorname{div}_x \mathbf{v} = 0$ yields $\partial f/\partial r + 2f/r = 0$, from which $f(t,r) = \beta(t)/r^2$, where $\beta(t)$ is an undetermined function. From (5.1) we have

$$r_0^3(t, R) = R^3 + 3b(t);\ r^3(t, \tau^*) = c_e^3(\tau) + 3(b(t) - b(\tau^*)) \tag{5.2}$$

where $c_e(t)$ is the outer and $c_i(t)$ the inner radius of the sphere at the time t (the remaining notation is analogous to the notation in Sec. 4).

For radial motion of the material points

$$\mathbf{e}_i(\mathbf{x})|_{\mathbf{x}=\chi(t,\xi)} = \mathbf{e}_i(\mathbf{X}(\xi)) = \mathbf{e}_i^*(\xi)$$
$$\mathbf{X}(\xi) = \chi(\tau^*(\xi), \xi)$$
$$\mathbf{e}_i(\mathbf{x})|_{\mathbf{x}=\chi_0(t,\mathbf{X})} = \mathbf{e}_i(\mathbf{X}),\quad \mathbf{X} \in \Omega_0$$

where $\mathbf{e}_i(\mathbf{x})$ are unit vectors of the basis of the spherical coordinate system. Taking account of the initial deformation $\mathbf{F}_*(\xi) = F_*(\tau^*)(\mathbf{e}_2^* \otimes \mathbf{e}_2^* + \mathbf{e}_3^* \otimes \mathbf{e}_3^*) + F_*^{-2}(\tau^*)\mathbf{e}_1^* \otimes \mathbf{e}_1^*$, we obtain

$$\mathbf{F}(t,\xi) = F(t,\tau^*)(\mathbf{e}_2^* \otimes \mathbf{e}_2^* + \mathbf{e}_2^* \otimes \mathbf{e}_3^*) + F^{-2}(t,\tau^*)\mathbf{e}_1^* \otimes \mathbf{e}_1^* \tag{5.3}$$
$$\mathbf{F}_0(t, X) = F_0(t, R)(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) + F_0^{-2}(t, R)\,\mathbf{e}_1 \otimes \mathbf{e}_1,\ \mathbf{e}_i = \mathbf{e}_i(\mathbf{X})$$
$$F(t, \tau^*) = F_*(\tau^*)\, r(t, r^*)/c_e(\tau^*)$$
$$F_0(t, R) = r_0(t, R)/R,\quad \mathbf{X} = \mathbf{X}(R, \Theta, Z)$$

An arbitrary dependence of the kernel μ on R is allowable in the governing Eqs.(4.10)-(4.11) for the original cylinder and on τ^* for the incremental elements. Also admissible for the latter is any preliminary deformation of the form

$$\mathbf{F}(\tau, \xi) = F_1(\mathbf{e}_2^* \otimes \mathbf{e}_2^* + \mathbf{e}_3^* \otimes \mathbf{e}_3^*) + F_1^{-2}\mathbf{e}_1^* \otimes \mathbf{e}_1^*$$
$$F_1 = F_1(\tau, \tau^*),\ t_0(\tau^*) \leqslant \tau \leqslant \tau^*$$

Under these conditions, we have for non-zero physical components of the stress tensor

$$\sigma_\varphi = \sigma_\lambda,\ \sigma(t, r) = F^2(t)\, L(t - t_0, [1 - F^{-2}I_1]^t) - F^{-4}(t)\, L(t - t_0, [1 - F^4 I_1]^t),\ I_1 = (2F^2 + F^{-4})/3 \tag{5.4}$$
$$\sigma = \sigma_\varphi - \sigma_r$$

For brevity, the zero superscript is omitted here on the quantities $F_0, \sigma_r^\circ, \sigma_\varphi^\circ, \sigma_\lambda^\circ$, that characterize points of the original sphere; also omitted is the argument τ^* (R, respectively) for the functions t_0 and F, and the inverse functions $\tau^*(t, r)$ and $R(t, r)$ inverse to (5.2) must be substituted in place of τ^* and R. Integrating the equilibrium equation $\partial_r \sigma_r = 2\sigma/r$, we obtain

$$\sigma_r^\circ(t, r) = -p_i(t) + V^\circ(t, r) - V^\circ(t, c_i(t)) \tag{5.5}$$
$$r \in [c_i(t), r_0(t, R_2)]$$
$$\sigma_r(t, r) = -p_e(t) + V(t, r) - V(t, c_e(t))$$
$$r \in [r(t, 0), c_e(t)]$$
$$V(t, r) = 2\int \frac{\sigma(t, r)\, dr}{r},\quad V^\circ(t, r) = 2\int \frac{\sigma^\circ(t, r)\, dr}{r}$$

The system of equations to find the undetermined functions $\beta(t)$, $c_i(t)$ and $c_e(t)$ include the continuity condition for the radial stress on the boundary of the original and accreted domains (see (5.5))

$$V^\circ(t, r_0(t, R_2)) - V^\circ(t, c_i(t)) + V(t, c_e(t)) - V(t, r(t, 0)) = (p_i - p_e)(t) \tag{5.6}$$

and the accretion ($J_e > 0$) and the reduction ($J_i > 0$)

$$\frac{dc_e}{dt} = J_e(t) + \frac{\beta(t)}{c_e^2(t)},\quad \frac{dc_i}{dt} = J_i(t) + \frac{\beta(t)}{c_i^2(t)} \tag{5.7}$$

This research was announced in /10/. After it had been sent to the editor, paper /11/ was published where another formulation is given of the problem of finite deformations of a growing body. The author is grateful to N.Kh. Arutyunyan for supporting the research.

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STABILITY OF BODIES MADE OF NON-HOMOGENEOUSLY AGING ANISOTROPIC, VISCOELASTIC MATERIAL *

V.D. POTAPOV

Results of the study of stability of compressed rods made of a non-homogeneously aging viscoelastic material are generalized to the case of an arbitrary body with anisotropy.

Let us consider a body acted upon by volume forces $\mathbf{F}$ and surface loads $\mathbf{q}$ applied at the boundary of the body S_q, in an orthogonal x_i $(i = 1, 2, 3)$, $\mathbf{F} = \{F_i\}$, $\mathbf{q} = \{q_i\}$ coordinate system. The points of the body undergo, under the action of these forces, the displacements $u_i(t, \mathbf{x})$ determining the trajectory of the unperturbed motion.

Let us assume that in the initial state the body has a small initial distortion $\alpha v_i^\circ(\mathbf{x})$. In this case the body undergoes additional displacements $\alpha v_i(t, \mathbf{x})$ so that the total displacement is $u_i^* = u_i + \alpha(v_i + v_i^\circ)$. The parameter α is introduced arbitrarily (and can be assumed equal to unity). The motion of the body determined by the displacements u_i^* will be called perturbed, and the displacements αv_i will be called perturbations.

Let us introduce the displacement norm (V is the volume of the body)

$$\|\mathbf{u}\| = \Big(\int\limits_V u_i u_i\, dV\Big)^{1/2}.$$

Here and henceforth the repeated indices denote summation.

Definition. An unperturbed motion of a viscoelastic body will be called stable, if for any number $A > 0$ a number $\delta = \delta(A) > 0$ can be found such that for any initial distortion αv_i° satisfying the inequality $\alpha\|\mathbf{v}^\circ\| < \delta$, the corresponding displacements αv_i satisfy the inequality $\alpha\|\mathbf{v}\| < A$, $0 \leqslant t < \infty$.

If the motion of the body is studied within a finite time interval $[0, T]$ and the critical value of the displacement norm $\|\mathbf{v}\|^*$ is given, we can speak of the critical time t_*, defining it as the instant at which the displacement norm $\alpha\|\mathbf{v}\|$ first attains the value $\|\mathbf{v}\|^*$: $\alpha \max\|\mathbf{v}(t)\| < \|\mathbf{v}\|^*$, $0 \leqslant t < t_*$ with $\alpha\|\mathbf{v}(t_*)\| = \|\mathbf{v}\|^*$.

We shall call the body stable in the time interval $[0, T]$, if $t_* > T$.

Analogous definitions of stability were used in connection with the non-homogeneously aging viscoelastic rods in [1, 2] where $\sup\limits_{t,x} |y(t, x)|$, $x \in [0, l]$ (l is the rod length) was used as the rod deflection norm.

Assuming that the deformations are small, we write the equations of state for the material in the form /1/

$$\sigma_{ij} = (E_{ijkl} - \mathbf{R}_{ijkl})\varepsilon_{kl} \tag{1}$$

$$E_{ijkl} = E_{ijkl}(t + \rho(\mathbf{x})), \quad \mathbf{R}_{ijkl}\varepsilon_{kl} = \int\limits_0^t R_{ijkl}^\rho \varepsilon_{kl}(\tau)\, d\tau, \quad R_{ijkl}^\rho = R_{ijkl}(t + \rho(\mathbf{x}), \tau + \rho(\mathbf{x}))$$

The moduli of elasticity E_{ijkl} and relaxation kernels R_{ijkl}^ρ of the material satisfy the following relations:

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